SINGULARITIES OF SOLUTIONS TO COMPRESSIBLE EULER EQUATIONS WITH VACUUM

Zhen Lei[1] and Yi Du[2] and Qingtian Zhang[3]

ABSTRACT. Presented are two results on the formation of finite time singularities of solutions to the compressible Euler equations in two and three space dimensions for isentropic, polytropic, ideal fluid flows. The initial velocity is assumed to be symmetric and the initial sound speed is required to vanish at the origin. They are smooth in Sobolev space H^3, but not required to have a compact support. It is shown that the H^3 norm of the velocity field and the sound speed will blow up in a finite time.

Keywords: Finite time singularities, compressible Euler equations, vacuum.

1. INTRODUCTION

Euler equation is one of the most fundamental equations in fluid dynamics. Many interesting fluid dynamic phenomena can be described by the Euler equation (see, for instance, [13]). Recently, singularity formation in fluid mechanics has attracted the attention of a number of researchers, see, for instance, [2, 3, 5, 7, 9, 8] and two recent review articles [1, 4]. For compressible Euler equations of the motion of polytropic ideal fluid flows, Sideris (see [14]) proved, under various settings, several very interesting results on the formation of finite time singularities to solutions whose initial velocity field has a compact support and initial density is strictly positive and is equal to a positive constant outside the support of the initial velocity field. It is very interesting to investigate the long time behavior of solutions to compressible Euler equations with initial data containing vacuum states, as has been pointed out in [11].

In this short article, we prove that solutions to compressible Euler equations will develop finite time singularities for radially symmetric initial data whose initial velocity field has no compact support and initial density contains vacuum states. In particular, it is shown that the H^3 norm of the velocity field and the sound speed will blow up in a finite time.

[1]School of Mathematical Sciences; LMNS and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, P. R. China. *Email: leizhn@gmail.com*

[2]School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P. R. China. *Email: duyidy@gmail.com*

[3]Department of Mathematics, Penn State University, State College, PA 16801, USA. *Email: zhang_q@math.psu.edu*

arXiv:1203.2396v2 [math.AP] 22 Mar 2012

To state our theorems, let us begin with the compressible Euler equations for isentropic, polytropic, ideal fluid flows:

$$(1.1) \qquad \begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p = 0, \end{cases}$$

where $\rho(t, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is the scalar mass density, $u(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is the velocity field, $p(t, x)$ is the pressure which is given by the equation of state

$$(1.2) \qquad p = A\rho^\gamma.$$

Here $A > 0$ is an entropy constant, γ is the adiabatic index. For polytropic gases, one has $1 < \gamma \le \frac{5}{3}$.

The compressible Euler equations (1.1) are imposed on the following initial data

$$(1.3) \qquad \rho(0, x) = \rho_0(r), \quad u(0, x) = \frac{x}{r} v_0(r).$$

Here and in what follows, we will use

$$r = |x|, \quad x \in \mathbb{R}^n,$$

for notational convenience. Denote the sound speed $c(\rho)$ by

$$(1.4) \qquad c(\rho) = \sqrt{\frac{\partial p(\rho)}{\partial \rho}} = \sqrt{A\gamma} \rho^{\frac{\gamma-1}{2}}.$$

Our first result is on the formation of singularities of solutions to the compressible Euler equations in three dimensions:

Theorem 1.1. *Assume that $\gamma > 1$, $\rho_0 \ge 0$ and $(c_0, u_0) \in H^3(\mathbb{R}^3)$ with $c_0 = c(\rho_0)$. Moreover, assume that ρ_0 and v_0 satisfy*

$$(1.5) \qquad \rho_0(0) = 0,$$

$$(1.6) \qquad \int_{\mathbb{R}^3} \rho_0(r) dx > 0,$$

and

$$(1.7) \qquad -\int_{\mathbb{R}^3} \frac{(1+r)\rho_0 v_0}{r^2 e^r} dx \ge \sqrt{\frac{A}{(\gamma+1)(4\pi)^{\gamma-1}}} \left(\int_{\mathbb{R}^3} \frac{\rho_0}{r e^r} dx \right)^{\frac{\gamma+1}{2}}.$$

Then the solution (ρ, u) to the compressible Euler equations (1.1) with the initial data (1.3) will develop finite time singularities.

Our second result is on the formation of singularities of solutions to the compressible Euler equations in two dimensions:

Theorem 1.2. *Assume that $\gamma > 1$, $\rho_0 \ge 0$ and $(c_0, u_0) \in H^3(\mathbb{R}^2)$ with $c_0 = c(\rho_0)$. Let $K_0(r)$ be the modified Bessel function*

$$(1.8) \qquad K_0(r) = \int_0^\infty e^{-r \cosh t} dt.$$

If the initial data ρ_0 and v_0 satisfy (1.5),

$$(1.9) \qquad \int_{\mathbb{R}^2} \rho_0(r)dx > 0,$$

and

$$(1.10) \qquad \int_{\mathbb{R}^2} \rho_0(r)v_0(r)K_0'(r)dx \geq \sqrt{\frac{A}{\gamma+1}} \frac{\left(\int_{\mathbb{R}^2} \rho_0(r)K_0(r)dx \right)^{\frac{\gamma+1}{2}}}{\left(\int_{\mathbb{R}^2} K_0(r)dx \right)^{\frac{\gamma-1}{2}}},$$

then the solution (ρ, u) to the compressible Euler equations (1.1) *with the initial data* (1.3) *will develop finite time singularities.*

Remark 1.3. *For a polytropic ideal gas, the adiabatic index $\gamma \in (1, \frac{5}{3}]$ and hence $\frac{2}{\gamma-1} \geq 3$. Due to the expression of the sound speed in* (1.4), *it is easy to get*

$$\rho_0 = \left(A\gamma\right)^{-\frac{1}{\gamma-1}} c_0^{\frac{2}{\gamma-1}}.$$

Consequently, one also has $\rho_0 \in H^3(\mathbb{R}^n)$ under the condition that $c_0 \in H^3(\mathbb{R}^n)$, which in turn implies that $\rho(t, \cdot) \in H^2(\mathbb{R}^n)$ as long as the solution is smooth. For $\gamma > \frac{5}{3}$, local well-posedness theory in Theorem 2.1 implies that $\rho \in C([0, T] \times \mathbb{R}^n)$.

Remark 1.4. *For the non-isentropic, polytropic, ideal gases, the entropy S is transported by the flows. It is easy to verify that our proofs of the blow up parts in Theorem 1.1 and Theorem 1.2 are still true. However, the local well-posedness of the coupled system with initial vacuum is much more complicated (see, for instance, [11]). We do not pursue this issue in this short article.*

There are several ingredients in the proofs of the above theorems. The first one is to write the compressible Euler equations (1.1) as a quasi-linear wave type equation in terms of ρ with inhomogeneous terms involving u:

$$\rho_{tt} - \Delta p = \nabla \cdot [\nabla \cdot (\rho u \otimes u)].$$

This is in fact what Sideris did in [14]. However, we will treat this equation in a very different manner from that in [14] due to the facts that $\rho(r) \to 0$ as $r \to \infty$ and the initial velocity field u_0 has no compact support. We will choose the modified Bessel function $K_0(r)$ in 2D case and $\frac{1}{re^r}$ in 3D case as test functions for the above quasi-linear wave type equation, respectively, to explore the nonlinear structure of the pressure p as a function of ρ, which eventually corresponds to the formation of finite time singularities. The second one is making use of the symmetric structure of the solutions, which results in a good sign for the inhomogeneous term $\nabla \cdot [\nabla \cdot (\rho u \otimes u)]$ when taking the inner product of the above wave type equation with test functions. Moreover, the symmetric structure of the solutions will also be used to eliminate boundary terms on the artificial boundary $r = 0$.

The remaining part of this paper is simply organized as follows: In section 2 we will prove Theorem 1.1. Then in section 3 we present the proof of Theorem 1.2.

2. SINGULARITIES OF COMPRESSIBLE EULER EQUATIONS IN 3D

In this section we will prove Theorem 1.1. Before that, let us recall the local well-posedness of compressible Euler equations (for example, see [12]). To be precise, let us formulate it as a theorem for our use.

Theorem 2.1. *Assume that $\gamma > 1$, $\rho_0 \geq 0$ and $(c_0, u_0) \in H^3(\mathbb{R}^n)$ with $c_0 = c(\rho_0)$ and $n = 2, 3$. Then there exists a unique solution (ρ, u) to the compressible Euler equations (1.1) with initial data (1.3) on some time interval $[0, T)$, which satisfies*

$$c(\rho), \; u \in C([0,T), H^3(\mathbb{R}^n)) \cap C^1([0,T), H^2(\mathbb{R}^n)) \cap C^2([0,T), H^1(\mathbb{R}^n)),$$

and

(2.1) $$\rho \in C([0,T) \times \mathbb{R}^n).$$

If $1 < \gamma \leq \frac{5}{3}$, then

(2.2) $$\rho \in C([0,T), H^2(\mathbb{R}^n)) \cap C^1([0,T), H^1(\mathbb{R}^n)).$$

If ρ_0 and u_0 are radially symmetric and (1.5) is satisfied, then

(2.3) $$\rho(t,0) \equiv 0, \; u(t,0) \equiv 0, \quad u(t,x) = \frac{x}{r}v(t,r).$$

The proof of the local well-posedness in the above theorem is based on the fact that the compressible Euler equations can be written as a symmetric hyperbolic system in terms of (c, u):

$$\begin{cases} c_t + u \cdot \nabla c + \frac{2c}{\gamma-1}\nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \frac{\gamma c}{\gamma-1}\nabla c = 0. \end{cases}$$

See [12] for more details. The fact (2.2) is due to Remark 1.3 and the equation of mass conservation in the original compressible Euler equations (1.1). The fact $u(t,0) \equiv 0$ is in fact a universal identity for smooth symmetric vector. To see $\rho(t,0) \equiv 0$, we use $u(t,0) \equiv 0$ and the equation of mass conservation to get

$$\rho(t,0) = \rho_0(0)e^{-\int_0^t \nabla \cdot u(s,0)ds}.$$

It is ready to present the proof of Theorem 1.1.

Proof. We prove Theorem 1.1 by contradiction. Suppose that the solution $(c, u) \in H^3(\mathbb{R}^3)$ for all time $t \geq 0$ and $T = \infty$ in Theorem 2.1. We will derive that the density blows up in the ball centered at the origin with an arbitrary small radius $r_0 > 0$ in a finite time, which contradicts with (2.1).

Applying the time derivative to the first equation of the compressible Euler system (1.1), we have (in the sense of distribution)

(2.4) $$\rho_{tt} = -\nabla \cdot (\rho u)_t = \Delta p + \nabla \cdot [\nabla \cdot (\rho u \otimes u)].$$

Taking the L^2 inner product of the above equation with the test function $\frac{1}{re^r}$, one has

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3} \frac{\rho}{re^r}dx = \int_{\mathbb{R}^3} \Delta p \frac{1}{re^r}dx + \int_{\mathbb{R}^3} \nabla \cdot [\nabla \cdot (\rho u \otimes u)]\frac{1}{re^r}dx.$$

Let us first compute that

$$\int_{\mathbb{R}^3} \Delta p \frac{1}{re^r} dx$$

$$= \int_{\mathbb{R}^3} p\Delta \frac{1}{re^r} dx - \lim_{\varepsilon \to 0} \int_{|x|=\varepsilon} \Big(\frac{\partial p}{\partial r}\frac{e^{-r}}{r} - \frac{d(\frac{1}{re^r})}{dr}p\Big)ds$$

$$= \int_{\mathbb{R}^3} p\frac{1}{re^r} dx + \lim_{\varepsilon \to 0} \int_{|x|=\varepsilon} \Big(\frac{1}{re^r}\Big)'pds.$$

Using the equation of state (1.2) and the fact that $\rho(t,0) \equiv 0$, we have

$$A\int_{\mathbb{R}^3} \Delta \rho^\gamma \frac{1}{re^r} dx = A\int_{\mathbb{R}^3} \rho^\gamma \frac{1}{re^r} dx$$

$$\geq \frac{A}{(4\pi)^{\gamma-1}}\Big(\int_{\mathbb{R}^3} \frac{\rho}{re^r} dx\Big)^\gamma.$$

On the other hand, noting (2.3), one has

$$\int_{\mathbb{R}^3} \frac{1}{re^r}\nabla \cdot [\nabla \cdot (\rho u \otimes u)]dx$$

$$= \int_{\mathbb{R}^3} \partial_i\partial_j\Big(\frac{1}{re^r}\Big)(\rho u_i u_j)dx$$

$$= \int_{\mathbb{R}^3} \rho v^2 \frac{x_j}{r}\partial_r\Big[\frac{x_j}{r}\Big(\frac{1}{re^r}\Big)'\Big]dx$$

$$= \int_{\mathbb{R}^3} \rho v^2 \Big(\frac{1}{re^r}\Big)''dx$$

$$= \int_{\mathbb{R}^3} \rho v^2 \Big[\frac{1}{r} + \frac{2}{r^2} + \frac{2}{r^3}\Big]e^{-r}dx > 0.$$

Here and in what follows, we use Einstein's convention for summation over repeated indices. Consequently,

$$(2.5) \qquad \frac{d^2}{dt^2}F(t) \geq \frac{A}{(4\pi)^{\gamma-1}}\big(F(t)\big)^\gamma,$$

where

$$F(t) = \int_{\mathbb{R}^3} \frac{\rho(t,r)}{re^r} dx.$$

Using the equation of conservation of mass and integration by parts, and noting (1.7), one has

$$(2.6) \qquad F'(0) = \frac{d}{dt}\int_{\mathbb{R}^3} \frac{\rho}{re^r} dx\Big|_{t=0} = -\int_{\mathbb{R}^3} \frac{(1+r)\rho_0 v_0}{r^2 e^r} dx > 0.$$

The combination of (2.5) and (2.6) gives that

$$F'(t) = F'(0) + \int_0^t \frac{d^2}{ds^2}F(s)ds \geq F'(0) > 0.$$

Consequently, one can multiply the both sides of (2.5) by $\frac{d}{dt}F(t)$ to get

$$\left(F'(t)\right)^2 \geq \frac{A}{(\gamma+1)(4\pi)^{\gamma-1}}\left(F(t)\right)^{\gamma+1}$$
$$+\left(\frac{d}{dt}\int_{\mathbb{R}^3}\frac{\rho}{re^r}dx\right)^2\Big|_{t=0} - \frac{A}{(\gamma+1)(4\pi)^{\gamma-1}}\left(\int_{\mathbb{R}^3}\frac{\rho_0}{re^r}dx\right)^{\gamma+1}.$$

Noting (1.7) and (2.6), and by denoting

$$C_0 = \sqrt{\frac{A}{(\gamma+1)(4\pi)^{\gamma-1}}},$$

one has

$$F'(t) \geq C_0\left(F(t)\right)^{\frac{\gamma+1}{2}},$$

which implies that

$$\int_{\mathbb{R}^3}\frac{\rho}{re^r}dx \geq \left(F(0)^{-\frac{\gamma-1}{2}} - \frac{\gamma-1}{2}C_0t\right)^{-\frac{2}{\gamma-1}}.$$

Noting (1.6), one has

$$\int_{\mathbb{R}^3}\frac{\rho_0}{re^r}dx > 0,$$

By using the mass conservation, one has

$$F(t) = \int_{\mathbb{R}^3}\frac{\rho}{re^r}dx \leq \int_{B_{r_0}}\frac{\rho}{re^r}dx + \frac{1}{r_0}\int_{\mathbb{R}^3}\rho_0 dx,$$

for any given $r_0 > 0$, here B_{r_0} is the three-dimensional ball centered at the origin with radius r_0. Consequently, one concludes that $\int_{r\leq r_0}\rho(t,r)rdr$ can not be bounded as $t \to \frac{2F(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$, which in turn implies that $\rho(t,r)$ will blow up for $r \leq r_0$ as $t \to \frac{2F(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$. By the assumption $c \in L^\infty([0,\infty), H^3(\mathbb{R}^3)$ and Sobolev imbedding theorem, one has $c \in L^\infty([0,\infty) \times \mathbb{R}^3)$ and hence $\rho \in L^\infty([0,\infty) \times \mathbb{R}^3)$. We arrive at a contradiction. So the H^3 norm of (c,u) can not be bounded before the time $\frac{2F(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$. The proof of Theorem 1.1 is completed. □

3. Singularities of compressible Euler equations in 2D

In this section we prove Theorem 1.2. First of all, let us recall that the modified Bessel function $K_0(r)$ and its derivative $K_0'(r)$ decay sufficiently fast as $r \to \infty$. In fact, for any

given $k > 1$, one has

$$\sup_{0<r<\infty} r^k K_0(r) = \sup_{0<r<\infty} r^k \int_0^\infty e^{-r\cosh t} dt$$

$$= \sup_{0<r<\infty} r^k \int_0^1 e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt + \sup_{0<r<\infty} r^k \int_1^\infty e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt$$

$$\leq \sup_{0<r<\infty} r^k e^{-\frac{r}{2e}} \int_0^1 e^{-\frac{re^t}{2}} dt + \sup_{0<r<\infty} \int_1^\infty r^k e^{-\frac{re^t}{2}} (\frac{r}{2} e^t)^k (\frac{2}{r} e^{-t})^k dt$$

$$\leq \sup_{0<r<\infty} r^k e^{-\frac{r}{2e}} + \sup_{0<r<\infty} 2^k \int_1^\infty e^{-kt} dt \sup_{0<s<\infty} e^{-s} s^k$$

$$< \infty$$

and

$$\sup_{0<r<\infty} r^k |K_0'(r)| = \sup_{0<r<\infty} r^k \int_0^\infty e^{-r\cosh t} \cosh t \, dt$$

$$= \sup_{0<r<\infty} 2r^k \int_0^1 e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt + \sup_{0<r<\infty} r^k \int_1^\infty e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} e^t dt$$

$$\leq \sup_{0<r<\infty} 2r^k e^{-\frac{r}{2e}} + \sup_{0<r<\infty} 2^k \int_1^\infty e^{-(k-1)t} dt \sup_{0<s<\infty} e^{-s} s^k$$

$$< \infty.$$

On the other hand, for $r > 0$, we also have

$$K_0(r) \leq \int_0^1 e^{-r\cosh t} dt + \int_1^\infty e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt$$

$$\leq 1 + \int_1^\infty e^{-\frac{re^t}{2}} dt \leq 1 + \frac{2}{r},$$

and

$$|K_0'(r)| \leq \int_0^1 e^{-r\cosh t} e^t dt + \int_1^\infty e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} e^t dt$$

$$\leq 2 + \int_1^\infty e^{-\frac{re^t}{2}} e^t dt$$

$$\leq 2 + \int_1^\infty 2(\frac{re^t}{2})^{-2} e^t dt \leq 2 + \frac{1}{2r^2}.$$

We in fact have proved the following lemma:

Lemma 3.1. *The modified Bessel function $K_0(r) = \int_0^\infty e^{-r\cosh t} dt$ satisfies*

$$\begin{cases} K_0(r) \leq \frac{3}{r}, |K_0'(r)| \leq \frac{1}{r^2}, & 0 < r < \frac{1}{2}, \\ K_0(r) \leq \frac{C_k}{r^k}, |K_0'(r)| \leq \frac{C_k}{r^k}, & r > 1, \end{cases}$$

for constants C_k depending only on $k > 1$.

Now it is ready to present the proof of Theorem 1.2.

Proof. We prove Theorem 1.2 by contradiction. Suppose that the solution $(c, u) \in H^3(\mathbb{R}^3)$ for all time $t \geq 0$ and $T = \infty$ in Theorem 2.1. We will derive that the density blows up in the ball centered at the origin with an arbitrary small radius $r_0 > 0$ in a finite time, which contradicts with (2.1).

Taking the L^2 inner product of the above equation with the test function $K_0(r)$, one has

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^2} \rho K_0(r) dx = \int_{\mathbb{R}^2} \Delta p K_0(r) dx + \int_{\mathbb{R}^2} \nabla \cdot [\nabla \cdot (\rho u \otimes u)] K_0(r) dx.$$

Using a similar argument as in section 2 and noting the decay properties of the Bessel function $K_0(r)$ in Lemma 3.1, we have

$$\int_{\mathbb{R}^2} \Delta p K_0(r) dx = \int_{\mathbb{R}^2} p \Delta K_0(r) dx.$$

Noting that the modified Bessel function $K_0(r)$ satisfies

$$(3.1) \qquad\qquad\qquad K_0'' + \frac{1}{r} K_0' = K_0,$$

one has

$$\int_{\mathbb{R}^2} \Delta p K_0(r) dx = A \int_{\mathbb{R}^2} \rho^\gamma K_0(r) dx$$

$$\geq \frac{A}{\left(\int_{\mathbb{R}^2} K_0(r) dx \right)^{\gamma - 1}} \left(\int_{\mathbb{R}^2} \rho K_0(r) dx \right)^\gamma.$$

On the other hand, noting (2.3) and using (3.1) and Lemma 3.1, one has

$$\int_{\mathbb{R}^2} K_0(r) \nabla \cdot [\nabla \cdot (\rho u \otimes u)] dx$$

$$= \int_{\mathbb{R}^2} \rho v^2 K_0''(r) dx$$

$$= \int_{\mathbb{R}^2} \rho v^2 \left(K_0(r) - \frac{1}{r} K_0'(r) \right) dx > 0.$$

Here we also used the fact that $K_0'(r) < 0$ by the expression of the modified Bessel function $K_0(r)$. Consequently, we have

$$(3.2) \qquad\qquad\qquad \frac{d^2}{dt^2} G(t) \geq \frac{A G(t)^\gamma}{\left(\int_{\mathbb{R}^2} K_0(r) dx \right)^{\gamma - 1}}.$$

where

$$G(t) = \int_{\mathbb{R}^2} \rho(t, r) K_0(r) dx.$$

Using the equation of conservation of mass and integration by parts, and noting (1.9), one has

$$(3.3) \qquad\qquad G'(0) = \frac{d}{dt} \int_{\mathbb{R}^2} \rho K_0(r) dx \Big|_{t=0} = \int_{\mathbb{R}^2} \rho_0 v_0 K_0'(r) dx > 0.$$

The combination of (3.2) and (3.3) gives that

$$G'(t) = G'(0) + \int_0^t \frac{d^2}{ds^2} G(s) ds \geq G'(0) > 0.$$

Consequently, one can multiply the both sides of (3.2) by $\frac{d}{dt}G(t)$ to get

$$\left(G'(t)\right)^2 \geq \frac{A}{(\gamma+1)\left(\int_{\mathbb{R}^2} K_0(r)dx\right)^{\gamma-1}}\left(G(t)\right)^{\gamma+1}$$
$$+\left(\frac{d}{dt}\int_{\mathbb{R}^2}\rho K_0(r)dx\right)^2\Big|_{t=0} - \frac{A\left(\int_{\mathbb{R}^2}\rho_0 K_0(r)dx\right)^{\gamma+1}}{(\gamma+1)\left(\int_{\mathbb{R}^2} K_0(r)dx\right)^{\gamma-1}}.$$

Noting (1.10) and (3.3), and by denoting

$$C_1 = \sqrt{\frac{A}{(\gamma+1)}}\left(\int_{\mathbb{R}^2} K_0(r)dx\right)^{-\frac{\gamma-1}{2}},$$

one has

$$G'(t) \geq C_1\left(G(t)\right)^{\frac{\gamma+1}{2}},$$

which implies that

$$\int_{\mathbb{R}^2}\rho K_0(r)dx \geq \left(G(0)^{-\frac{\gamma-1}{2}} - \frac{\gamma-1}{2}C_1 t\right)^{-\frac{2}{\gamma-1}}.$$

Noting (1.9), one has

$$G(0) = \int_{\mathbb{R}^2}\rho_0 K_0(r)dx > 0,$$

By using the mass conservation, one has

$$G(t) = \int_{\mathbb{R}^2}\frac{\rho}{re^r}dx \leq \int_{B_{r_0}}\rho K_0(r)dx + \frac{\max_{r\geq r_0} K_0(r)}{r_0}\int_{\mathbb{R}^2}\rho_0 dx,$$

for any given $r_0 > 0$, here B_{r_0} is the two-dimensional ball centered at the origin with radius r_0. Consequently, one concludes that $\rho(t,r)$ will blow up for $r \leq r_0$ as $t \to \frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$. By the assumption $c \in L^\infty([0,\infty), H^3(\mathbb{R}^2))$ and Sobolev imbedding theorem, one has $c \in L^\infty([0,\infty)\times\mathbb{R}^2)$ and hence $\rho \in L^\infty([0,\infty)\times\mathbb{R}^2)$. We arrive at a contradiction. So the H^3 norm of (c,u) can not be bounded before the time $\frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$. The proof of Theorem 1.2 is completed.

$\square$

References

[1] Bardos, C.; Titi, E. S.; Euler equations for incompressible ideal fluids, *Uspekhi Mat. Nauk*, **62(3) 375** (2007), 5–46.

[2] Castro; Cordoba A. D.; Fefferman C.; Gancedo F.; Gomez-Serrano, M.; Finite time singularities for the free boundary incompressible Euler equations, (2011), arXiv:1112.2170v1.

[3] Constantin, P.; Note on loss of regularity for solutions of the 3D incompressible Euler and related equations, *Commun. Math. Phys.* **104** (1986), 311–326.

[4] Constantin, P.; On the Euler equations of incompressible fluids, *Bull. Amer. Math. Soc.*, **44** (2007), 603–621.

[5] Constantin, P.; Fefferman, C.; Majda, A.; Geometric constraints on potentially singular solutions for the 3-D Euler equation, *Comm. in PDEs.* **21** (1996), 559–571.

[6] Cordoba, D.; Fefferman, C.; On the collapse of tubes carried by 3D incompressible flows. *Comm. Math. Phys.* **222** (2001), 293–298.

[7] Coutand, D.; Shkoller, S.; On the finite-time splash and splat singularities for the 3-D free-surface Euler equations, (2012), preprint, avaliable online at arXiv:1201.4919v2.

[8] Hou, T. Y.; Lei, Zhen; On the stabilizing effect of convection for three dimensional incompressible flows. *Comm. Pure Appl. Math.* **62 (4)** (2009), 501–564.

[9] Hou, T. Y.; Lei, Zhen; *Finite time blow-up of a 3D Model for incompressible Euler equations.* Preprint, 2012.

[10] Hou, T. Y.; Shi, Z.; Wang, S.; On singularity formation of a 3D model for incompressible Navier-Stokes equations. *Adv. Math.* (2012), accepted for publication.

[11] Jang, J.; Masmoudi, N.; Well-posedness of compressible Euler equations in a physical vacuum, arxiv.org/pdf/1005.4441

[12] Kato, T., The Cauchy problem of the quasi-linear symmetric hyperbolic systems, Archive Ration. Mech. Anal., **58** (1975), 181–205.

[13] Lions, P. L.; Mathematical Topics in Fluid Mechanics: Compressible Models, *Oxford Lecture Series in Mathematics and Its Applications*, **2** (1998), no. 10, Oxford Univ Press.

[14] Sideris, T.C.; Formation of singularities in three-dimensional compressible fluids, *Comm. Math. Phy.*, **101** (1985), 475–485.